Connections: Vol II

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There is a natural notion of vertical tangent vectors of our bundle: if we work in a trivialization (x^i, a^j), then

$$V_p = \left\langle \left(\frac{\partial}{\partial a^j} \right)_{p=(x,a)} \right\rangle_{j=1,\cdots,r} = \text{Ker } \mathrm{d}\pi$$

are naturally "vertical" tangent vectors. We would like a notion of "horizontal" tangent vectors, i.e. a horizontal subspace $S_p \subset T_pE$ such that $T_pE = V_p \oplus S_p$. If $(a^j) = 0$, then we can naturally identify

$$T_pE \ni \left(\frac{\partial}{\partial x^i} \right)_{p=(x,0)} \cong \left(\frac{\partial}{\partial x^i} \right)_{\pi(p)=x} \in T_{\pi(p)}B$$

hence then there is a canonical splitting $T_pE = V_p \oplus S_p$. For $(a^j) \neq 0$ this doesn't work, since under a transition function $\psi_{\beta\alpha}(x)$ the former changes with a term $a\frac{\partial \psi}{\partial x}\frac{\partial}{\partial a}$. The conceptual reason is that the $\frac{\partial}{\partial x}$ are tangent to the chosen basis for our sections (cf figure 2 in other PDF).

Analogously to $V_p = \text{Ker } \mathrm{d}\pi$, we can define S_p as the kernel of a map, i.e. a projection

$$\mathrm{pr} : T_pE \to E_{\pi(p)}, \quad \text{given by} \quad e_j \otimes (\mathrm{d}a^j + A^j_k \mathrm{d}x^k)$$

which is the thing of real importance anyway. But in accordance with the usual treatment, let's say we simply pick a horizontal subspace S_p. Note it's of dimension $n = \dim(B)$. By linear algebra[1] we can say

$$S_p = \cap_{i=1}^r \text{Ker } \theta^i \quad \text{where} \quad \theta^i \in T_p^*E.$$

We can write $\theta^i = \alpha_{ij}\mathrm{d}a^j + \beta_{ik}\mathrm{d}x^k$. Moreover, α_{ij} is invertible: $\frac{\partial}{\partial a} \notin S_p$ hence for every j there is an i such that $\theta^i\left(\frac{\partial}{\partial a^j}\right) \neq 0$, hence α_{ij} has no zero eigenvalues. Multiplying θ^i with invertible matrices doesn't change S_p, hence w.l.o.g. $\theta^i = \mathrm{d}a^i + A^i_k(x,a)\mathrm{d}x^k$.

We now **assume** $A^i_k(x,a) = A^i_{jk}(x)a^j$ (it is not necessary, but gives nice and natural properties). Hence we see our S_p is fixed by this A^i_{jk} which can be shown to transform like a connection ought to. It is the same one appearing above in "pr". In fact:

$$\mathrm{pr} = \sum_{i=1}^r e_i \otimes \theta^i : T_pE \to E_{\pi(p)}$$

[1] An n-dimensional subspace of $\mathbb{R}^{n+r}$ is determined by $\cap \text{Ker}\theta^i$ where θ^i are $i = 1, \cdots, r$ elements in $(\mathbb{R}^{n+r})^*$.

and indeed Ker pr $= \cap_i$Ker θ^i.

We can define a covariant derivative

$$d_A s := \mathrm{pr} \circ \mathrm{d}s : T_b B \to E_b$$

which indeed satisfies the Leibniz rule. In fact: using the axiomatic definition of the covariant derivative, one can show that they are all of this form.

We say a section $s : B \to E$ is **parallel** or **covariant constant** if $d_A s = 0$. From the above definition, we can see this is equivalent to saying $ds(T_b B) \subset S_{s(b)}$, i.e.: the tangent spaces to s are horizontal.

How does this work out in components? On the one hand we know *the horizontal vectors* are those generated by

$$\boxed{S_p = \left\langle \frac{\partial}{\partial x^j} - A^i_{kj} a^k \frac{\partial}{\partial a^i} \right\rangle_{j=1,\cdots,n}}$$

since

$$e_l \otimes \left(\mathrm{d}a^l + A^l_{sk} a^s \mathrm{d}x^k\right) \left(\frac{\partial}{\partial x^j} - A^i_{kj} a^k \frac{\partial}{\partial a^i}\right) = 0 \ .$$

On the other hand, *the tangent space of* $s(B)$ *at* $s(b)$ is

$$\boxed{T_{s(b)} s(B) = \left\langle \frac{\partial}{\partial x^j} + \frac{\partial s^i}{\partial x^j} \frac{\partial}{\partial a^i} \right\rangle} \ .$$

But since $d_A s = 0$, we know

$$\frac{\partial s^i}{\partial x^j} = -A^i_{kj} s^k \ .$$

This tells us indeed that

$$\boxed{T_{s(b)} s(B) = S_{s(b)}} \ .$$

Similarly (and for this we don't even need the notion of covariant derivative!), if we have a path $\gamma : I \to B$, we can lift it to a path $\tilde{\gamma} : I \to E$ by demanding that the tangent space of $\tilde{\gamma}(I)$ is flat (at every point). Since now

$$T_{\tilde{\gamma}(t)} I = \left\langle \dot{x}^j \frac{\partial}{\partial x^j} + \dot{a}^i \frac{\partial}{\partial a^i} \right\rangle$$

if we want $T_p I \subset S_p$ for all $p = \gamma(t)$, that is equivalent to demanding

$$\dot{a}^i + A^i_{jk} a^j \dot{x}^k = 0$$

which has a unique solution given an initial element $\in E_{\gamma(0)}$. This is called the **parallel transport** of our vector along γ.